Math Circle
University of Arizona

Cardinality

Section 1 : Inroduction

In mathematics, we seek a formal way to quantify the size of a set. The notion of the size of set is called *cardinality*. Before we begin, we need to establish a few definitions.

Definition: Functions
A function $f : X \to Y$ consists of three objects

- A set of inputs X called a **domain**
- A set of outputs Y called a **codomain**
- A **rule** that assigns each input x to a single output denoted $f(x)$

We call f *surjective* or *onto* if for every element y in the codomain, there is an x in the domain such that $f(x) = y$ (i.e. the range of f is all of Y). We call f *injective* or *one-to-one* $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ (i.e. f sends different inputs to different outputs). Functions that are both injective ans surjective are called *bijective.*

Problem 1: *Draw what it means for a function f to be injective, surjective, and bijective? Try drawing a diagram that matches elements of a domain to a codomain satisfying each of these properties. Can you write some examples of these functions?*

The cardinality of a finite set is the number of elements that it contains. How do we make this idea rigorous?

Definition: Cardinality for Finite Sets
A set X is called *finite* if there is a bijection between X and the set $\{1, 2, \dots, n\}$. In this case, we say the cardinality of X is n and write $|X| = n$.

Problem 2: *Show that the cardinality of a finite set X is unique. That is, show that there is a unique n such that there is a bijection between X and $\{1, \dots, n\}$. (HINT: Suppose $m \neq n$ but that X has a bijection with both $\{1, \dots, n\}$ and $\{1, \dots, m\}$. Use this to derive a contradiction).*
Problem 3: *Let X be any set and $\mathcal{P}(X)$ be the set of all subsets of X. If $f : X \to \mathcal{P}(X)$ is a function, show this cannot be a subjection. (HINT: Consider the set of all of all x in the domain such that x is not in the subset $f(x)$).*

Section 2 : Cardinality on General Sets

For finite sets, the concept of cardinality is just counting the number of elements. But what about when the sets are infinite? The following thought exercise is intended to show the paradoxes of working with infinite sets.

Problem 4: *Imagine you operate a hotel with an infinite number of rooms labeled with room numbers 1, 2, 3, ... Suppose that each room is currently occupied when a new guest arrives. You cannot make a new room, but you can rearrange the current occupants. How do you accommodate the new guest?*
Problem 5: *Now instead suppose an infinitely long bus arrives with an infinite number of new guests.*

How do you accommodate your new guests? Try to only move each guest once.
Problem 6: *Now instead instead of a single infinitely long bus, suppose an infinite number of busses arrive, each carrying an infinite number of people. How do get each person a room while only moving each guest once?*

The point of this though exercise known as Hilbert's Hotel is to show that when working with infinite sets, it's possible to map an infinite set into a smaller subset. For instance one can map the all room numbers in Hilbert's Hotel bijectively. The room numbers in Hilbert's hotel can be mapped to smaller subsets.

Definition: Cardinality for General Sets
Let X and Y be sets. We say that X and Y have the same cardinality, denoted by $|X| = |Y|$, if there exists a bijection $f : X \to Y$. We write $|X| \leq |Y|$ if there is an injection $f : X \to Y$. We write $|X| \geq |Y|$ if there exists a surjection $f : X \to Y$.
Definition: Countably Infinite Sets
Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of all positive counting numbers which we call the natural numbers. A set X is called countably infinite or countable if $|X| = |\mathbb{N}|$.

Problem 7: *Show that the set of all whole positive and negative counting numbers*

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\},$$

also known as the integers, is countable.
Problem 8: *The set of all rational numbers, denoted $\mathbb{Q}$, is the set of all fractions of whole numbers. Show that $\mathbb{Q}$ is countable. (HINT: First show that $|\mathbb{N}| \leq |\mathbb{Q}|$ and then separately that $|\mathbb{N}| \geq |\mathbb{Q}|$).*

The qualifier of "countable" infinity may suggest that there are larger infinities that are not countable. Indeed, some of the most fundamental objects that we take for granted are uncountable.

Problem 9: *Here we intend to show that the real numbers (the set of all numbers containing both numbers that are rational/fractions and all of the irrational numbers in between like $\sqrt{2}$), denoted $\mathbb{R}$, are uncountable. We will set our sights at a more modest target first ans show that the set of numbers between 0 and 1, denoted $(0,1)$ are uncountable. Let $f : \mathbb{R} \to (0,1)$ be a function. We can list the outputs of this function in a list:*

$$\begin{aligned} f(1) &= 0.a_1^1 a_2^1 a_3^1 \ldots \\ f(2) &= 0.a_1^2 a_2^2 a_2^2 \ldots \\ f(3) &= 0.a_1^3 a_2^3 a_3^3 \ldots \\ &\vdots \end{aligned}$$

Here each output is written as an infinitely long decimal where a_i^n is the i-th digit of $f(n)$. Can you generate an element of $(0,1)$ that is not on this list?
Problem 10: *Show that $(0,1)$ is uncountable. (HINT: Use number 9 to conclude that any $f : \mathbb{R} \to (0,1)$ cannot be a bijection).*
Problem 11: *Show that $|(0,1)| = |(0,2)| = |(0,1000000)|$.*
Problem 12: *Show that $|(0,1)| = \mathbb{R}$ and conclude that $\mathbb{R}$ is uncountable.*

Questions about cardinality don't need to be restricted to just numbers.

Problem 13: *Let the circle S^1 be the set of all points in the plane (x, y) with $x^2 + y^2 = 1$. Show that $|\mathbb{R}| = S^1$. Can you construct a bijection? (HINT: Try to use some geometry)*
Problem 14: *Let i denote the imaginary unit, a number that satisfies $i^2 = -1$ (think $\sqrt{-1}$). The complex numbers, denoted $\mathbb{C}$, are numbers of the form $a + bi$ where a and b are real numbers. Show that $\mathbb{C}$ is uncountable.*
Problem 14: *Let $\mathbb{A}$ denote the set of all algebraic numbers, i.e. all of the roots (including complex roots) of polynomials with rational coefficients. Show that $\mathbb{A}$ is countable. (HINT: First show that the collection of all polynomials with rational coefficients is countable. Can you factor these polynomials?)*
Problem 15: *The transcendental numbers are real numbers that are not algebraic. Show that the transcendental numbers are uncountable.*