Math Circle
University of Arizona

Graph Coloring

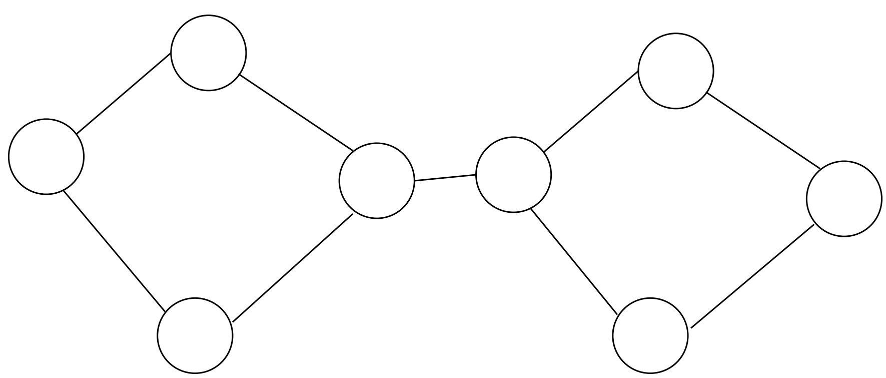
In mathematics, the fascinating field of graph theory deals with studying graphs, which are mathematical structures used to model pairwise relations between objects. A **graph** consists of **vertices** (or nodes) connected by **edges**. For our discussion, graphs adhere to a few rules:

1. Each edge connects two unique vertices, ensuring distinct pathways.
2. Each pair of vertices is connected by at most one edge, avoiding multiple connections.
3. Our graphs are **connected**, meaning there is a path between any two vertices, ensuring no isolated sections.

Graph **coloring** is a way to assign colors to each vertex so that no adjacent vertices (those directly connected by an edge) share the same color. A graph can be said to be k-colorable if it can be colored with k colors, with the minimum k being the graph's **chromatic number**. Graph coloring is widely used in various fields such as:

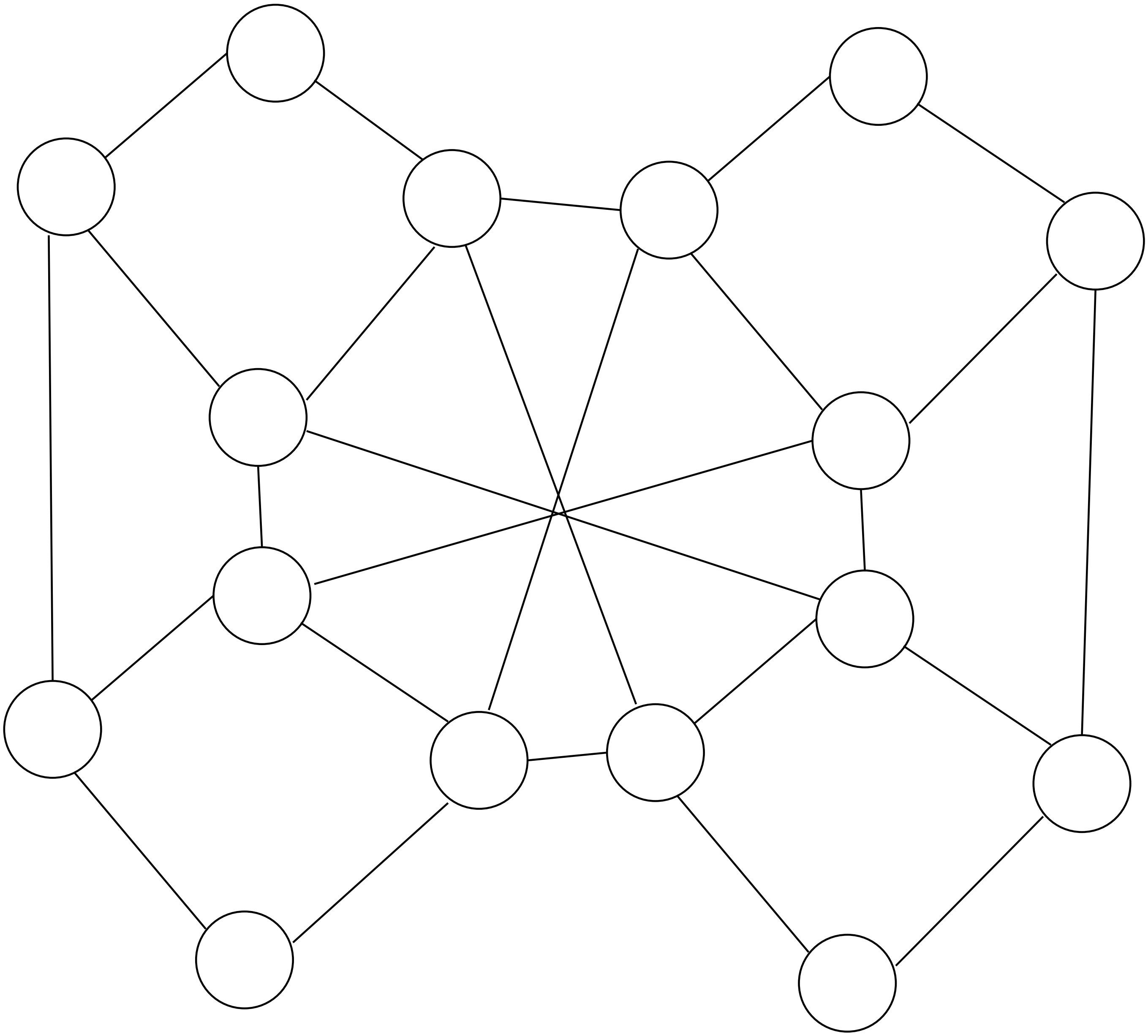
- Scheduling problems, where tasks must be assigned to time slots without conflicts.
- Register allocation in compilers, where variables are assigned to a limited set of registers.
- Map coloring, where adjacent regions must be colored differently to avoid confusion.

Problem 1: *Can you color this graph with 2 colors so no two vertices connected by an edge share the same color?*



Problem 2: *Consider a school that offers five different courses: Math, Physics, Chemistry, Biology, and Computer Science. Each course must be scheduled in such a way that no student has overlapping classes. Students in Mathematics typically also take Physics and Computer Science. Those in Physics,Biology, and Computer Science typically take Math courses. Those in Chemistry take Physics and Biology. How many time-slots are needed to so all students can take the courses they would like?*

Problem 3: *Consider a complex network of intersections (vertices) in a city. The main cause of the traffic is two neighboring intersections sharing the same schedule. To keep traffic at a minimum, the city can place the traffic lights on different schedules. How many different traffic light schedules are needed? (What is the minimum number of colors needed so that no two connected cities share the same color?)*



As we explore the practical side of graph coloring, it's clear that it's not just about solving puzzles. In math, we often look for the best ways to organize or schedule things, given a certain amount of recources. It is useful to formulate it as a graph coloring problem is why we often study graphs and their coloring in abstraction. Graph **coloring** helps us assign colors to each vertex, which in turn helps us figure out the least amount of resources we need, whether that's time slots, memory spaces, or colors on a map.

Problem 4: *Design connected graphs with chromatic numbers 3, 4, and 5. Explore how the graph's structure impacts its chromatic number and the strategies for coloring it.*

Problem 5: *Create a graph with a chromatic number between 2 and 5. Challenge a partner to determine its chromatic number, illustrating the process of analysis and deduction used in graph coloring.*

Problem 6: *A complete graph is one where every vertex connects to every other vertex. Determine the chromatic number of a complete graph with* n *vertices.*

Problem 7: *With configurations of 4 nodes and 4 edges, and then 5 nodes and 5 edges, create connected graphs. Investigate whether such graphs maintain the same chromatic number across different structures, highlighting the diversity of graph configurations.*

Problem 8: *With intuition gained, analyze connected graphs with n nodes and n edges to determine the minimum and maximum chromatic numbers possible. This exploration will deepen your understanding of the relationship between graph structure and chromatic number.*

Algorithms for Graph Coloring

Determining the chromatic number of a general graph is a well-known problem in computer science and discrete mathematics, characterized by its computational complexity. Specifically, the task of finding the exact chromatic number for any given graph is classified as an **NP-complete** problem. An NP-complete problem is one that is both in NP (nondeterministic polynomial time) and NP-hard. This means that while a solution to the problem can be verified quickly (in polynomial time), there is no known efficient way to find a solution for all possible cases. For graph coloring, this indicates that although we can quickly check if a given coloring of a graph using k colors is valid, discovering the minimum k (the chromatic number) can be exceedingly difficult as the size of the graph grows.

Approximating the Chromatic Number

Given the challenges associated with finding the exact chromatic number, researchers and practitioners often seek to approximate it. For many applications, finding a chromatic number that is close to the minimum can be sufficiently useful and can be achieved much faster than determining the exact value. Various heuristic algorithms exist for this purpose, providing good approximations in a fraction of the time required for an exact solution.

In practical scenarios, such as scheduling, network design, and resource allocation, an approximation of the chromatic number can still offer significant benefits. By reducing conflicts and optimizing the use of resources, even a non-minimal coloring solution can lead to improved operational efficiency and effectiveness.

Problem 9: *Develop an algorithm to find the approximate chromatic number of a graph. Test your algorithm on various graphs with known chromatic numbers to check how close you are.*

1. **Step 1**: Color node 1
2. **Step 2**:

Planar Graphs and Color Theorems

A **planar graph** is a graph that can be drawn on a plane without any of its edges crossing each other, except at their vertices. This unique property makes planar graphs a subject of considerable interest in both theoretical and applied graph theory.

The Four and Five Color Theorems

The study of planar graphs has led to two significant theorems regarding graph coloring:

- **The Four Color Theorem** states that any planar graph can be colored with no more than four colors in such a way that no two adjacent regions (faces) share the same color. Proving this theorem was a monumental task in mathematics, requiring computer assistance to check numerous cases, making it one of the first major theorems to be proved using a computer.
- **The Five Color Theorem** is an earlier result that is easier to prove using traditional mathematical methods. It asserts that five colors are sufficient to color any planar graph. Although it gives a less tight bound compared to the Four Color Theorem, its proof is accessible and provides valuable insights into graph coloring techniques.

While the Four Color Theorem is more famous for its tighter bound and the complexity of its proof, the Five Color Theorem offers a great opportunity to understand the principles behind graph coloring without delving into computational proofs.

Problem 10: *Attempt to create planar graphs with chromatic numbers 3, 4, and 5. While it's known that you won't need more than four colors for any planar graph, this exercise helps illustrate the concept of chromatic number in a tangible way. (Hint: Don't invest too much time trying to find a graph that requires 5 colors, as the Four Color Theorem assures us it doesn't exist.)*

Problem 11: *Explore the Five Color Theorem by attempting to prove it. This theorem, stating that any planar graph can be colored with five or fewer colors, offers a more approachable challenge than the Four Color Theorem and serves as a stepping stone in understanding graph coloring principles.*