Math Circle
University of Arizona

An Introduction to Kinematics and Dynamics!

1 Kinematics: The topology of motion!

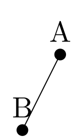
Definition 1. *A* ***configuration space*** $\mathcal{C}$ *is the set of all possible configurations of a mechanical system. Each configuration is described by a set of coordinates, the number of which is referred to as the* ***degrees of freedom (DOF)*** *of the system.*

For example, consider a point (particle) constrained to move on a line. If we identify this line with the real number line, choosing some arbitrary points to be 0 and 1, the position of the particle can be described with a *single coordinate*, a number expressing its distance from 0. The degrees of freedom is thus 1, and the configuration space of the particle is the line itself—that is, $\mathcal{C} = \mathbb{R}$.

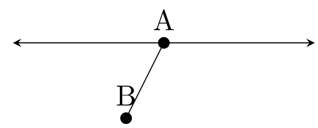
1.1. *Consider a point particle free to move anywhere in the plane. What are the degrees of freedom of the system, n? How can you represent the configuration space $\mathcal{C}$ for the particle?*

1.2. *Consider two point particles constrained to move on a line but able to pass through each other. What is n? How can you represent $\mathcal{C}$ for the system? How does this relate to the configuration space of problem 1? How would $\mathcal{C}$ change if the particles were not able to pass through each other?*

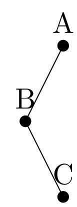
1.3. *Consider a point particle A that is fixed in space, and a second point particle B this is constrained to be a fixed distance from A. This system represents a pendulum, with A referred to as the pivot. We can also refer to the line of fixed length, AB, as a link. What is n? How can you represent $\mathcal{C}$ for the system?*



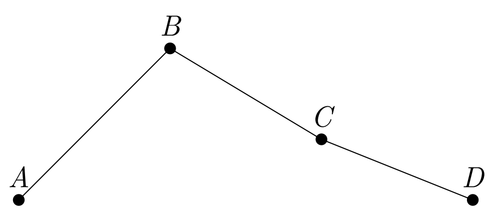
1.4. *Consider two point particles, A and B. A is constrained to move on a line, and B is constrained to be a fixed distance from A, but is otherwise free to move in the plane containing the line. What is n? How can you represent $\mathcal{C}$?*



1.5. *Consider two links, AB and BC, with A a pivot. This configuration is referred to as a double pendulum. What is n? How can you represent $\mathcal{C}$?*



1.6. *Consider three links, AB, BC, and CD, with A and D pivots. What is n? Does it depend on the lengths of the links?*



Take the distance between A and D to be 1, and the lengths of AB, BC, and CD to each be 1. How can you represent $\mathcal{C}$?

Take the distance between A and D to be 1, and the lengths of AB, BC, and CD to each be 0.5. How can you represent $\mathcal{C}$?

2 Dynamics: The determination of trajectories!

Definition 2. *Consider a configuration space C and an interval $I \subset \mathbb{R}$. A continuous, smooth curve $x : I \to C$ is referred to as a* ***trajectory****.*

For example, consider again the point particle constrained to move on a line, which we identified as its configuration space. Choosing the interval to be $I = [0, 1]$, a potential trajectory may be given by $x(t) = t$, so that the particle moves from 0 to 1 in 1 unit of time. An alternate trajectory could be $x(t) = \sin(\pi t)$, so that the particle moves from 0 to 1 and back to 0 in 1 unit of time.

In practice, we are not usually interested in arbitrary trajectories, but rather in the *natural* trajectories of a system. These are the trajectories that the system will follow if $n > 0$. But how do we go about determining these natural trajectories? That's where physics comes in!

It turns out nature has plans for uncontrolled degrees of freedom. In the systems we'll look at here, the natural trajectories are determined by the paths of constant*energy*, a function E of coordinates and velocity. Energy does not change over time for what are called *conservative* systems. Thus, conservative systems with $n = 1$ have natural trajectories that are fully determined by the conservation of energy.

An example of a conservative system is a ball of mass m falling straight down under the influence of gravity and in the absence of air resistance. For this system, the energy is given by the function

$$E(v, y) = \frac{1}{2}mv^2 + mgy. \tag{1}$$

The energy is the same at every point in the natural trajectory. What this means is the speed of the ball is just a function of its elevation! If the ball starts from rest from a height h, the fact that E is constant means

$$v(y) = \sqrt{2g(h - y)}. \tag{2}$$

(Make sure you can derive this from equation (1)!)

Let's try to calculate at what time T a ball will hit the ground ($y = 0$), assuming it is dropped from rest at a height $y = h$. If we knew something about differential equations, we could solve (2) directly to find an answer! But let's see if we can find the answer through other means. We see from (2) that as we get closer to the ground ($y \to 0$), the speed of the ball increases, reaching a maximum of $v_f = \sqrt{2gh}$ at $y = 0$. Were the ball traveling at this speed the entire time, it would take $T = \frac{h}{v} = \sqrt{\frac{h}{2g}}$ to hit the ground. However, we know that the ball starts from rest, so it must accelerate to this speed, and the real time $T > \sqrt{\frac{h}{2g}}$.

It is possible to compute T exactly using some mild calculus[1], but here we'll just illustrate how you might go

[1] Calculus students: If we differentiate (2) with respect to t, we find that

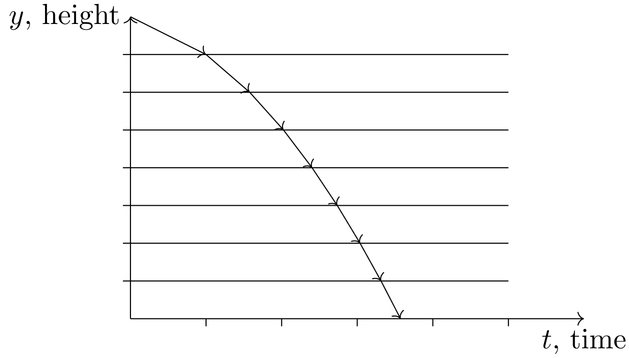
$$a(t) = \frac{dv}{dt} = \frac{1}{2}\sqrt{2g}\frac{(-dy/dt)}{\sqrt{h - y}} = -g\frac{v}{v} = -g. \tag{3}$$

This tells us the acceleration a is constant, and equal to g (the negative sign comes from the fact that velocity is directed downwards). The fact that acceleration is constant means velocity increases the same amount for each interval of time. We can then use the fact that

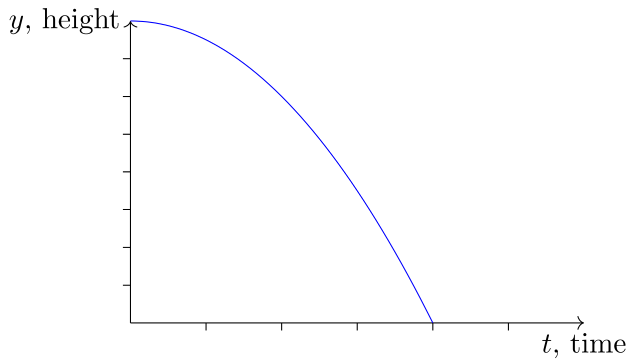
$$\int_0^T v(t)dt = \int_0^T (at)dt = gT^2/2 = h \tag{4}$$

to calculate T.

about finding T using a graph constructed from $v(y)$.



At each level, the function $v(y)$ gives us the speed of the ball; that is, how much the elevation will change in an instant of time. In our graph, $v(y)$ is giving us the slope at each level. If we break up the height into finer and finer intervals, we'll get closer and closer to the actual trajectory of the ball: in the limit, it looks like this:



which is a parabola! Comparing these two graphs, we can there's some error in our discrete approximation, but they still look similar!

2.1. *How can you come to the conclusion that* $T = \sqrt{2h/g}$ *from our graph?*

What if the ball was free to move in the plane, or even in three dimensions? Now $n = 2$ (or $n = 3$), and the natural trajectories are no longer determined by the conservation of energy alone. However, we can use another fact about mechanics: Galilean invariance! What Galilean invariance says is that the laws of physics are the same in all *inertial reference frames.*

Definition 3. *An* ***inertial reference frame*** *is a coordinate system in which Newton's first law holds: an object at rest will remain at rest, and an object in motion will remain in motion at a constant velocity, unless acted upon by a force.*

Considering again the particle constrained to move on a line, if the particle is initially moving with some speed, then the particle will continue to move with that speed. If we instead consider a reference frame moving at a constant velocity with the particle, the particle will initially be at rest, and will continue to stay at rest. Both of these reference frames are consistent with Newton's first law, and are thus inertial reference frames! However, if we instead shook the reference frame back and forth, we would introduce a fictitious force, causing the particle to move back and forth despite the absence of any real force.

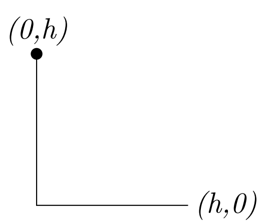
An example of an inertial reference frame is a frame translating at a constant velocity. When you fly on an airplane, you can toss a ball up and down, and it won't fly to the back of the cabin, or up towards the front! However, your ball tossing is thwarted as the plane lands and brakes to slow down, because the decelerating plane is no longer an inertial reference frame. In general, given some inertial reference frame F, if we apply some fixed translation, rotation, or constant velocity to F, we get another inertial reference frame, F': otherwise, we get a non-inertial reference frame haunted by spooky "fictitious" forces, like the one causing the ball to fly to the front of the plane.

How can we use Galileo's observation of inertial reference frames to determine the path for a ball falling in the plane [2]? If the ball is initially moving with some velocity, we can always find an inertial reference frame where the ball is initially at rest! However, the vertical part of the velocity we impose will cause the ground to appear to move at a constant speed, so we'll need to account for this when determining the collision time! This means that the horizontal component of the initial velocity won't affect our collision time.

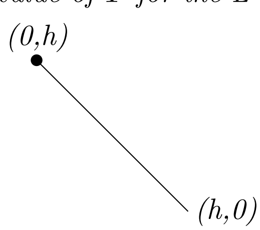
2.2. *How will the vertical component of the initial velocity affect the collision time?*

Next, let's consider the situation where the object is not allowed to take the natural parabolic path, but is instead constrained to move along a particular path at its natural speed. See if you can solve the following problems using our results from above!

2.3. *Consider a bead constrained to slide along a nearly L-shaped wire under the influence of gravity (nearly L-shaped, as the "angle" of the L is actually smooth). What is n? What is the configuration space of the bead? Can you show that the time when the bead reaches the end of the wire $(h, 0)$ after being released from rest at $(0, h)$ is $T = (\sqrt{2} + 1/\sqrt{2})\sqrt{(h/g)}$?*

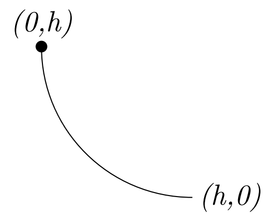


2.4. *A straight line is the path of shortest distance between two points in Euclidean space. Consider a bead constrained to slide along a straight wire under the influence of gravity. What is n? What is the configuration space of the bead? Can you show that $T = 2\sqrt{(h/g)}$ in this case? Is this value larger or smaller than the value of T for the L-shaped wire? Is this what you expected?*



[2] $\mathbb{R}^2$, not an airplane!

2.5. *Galileo comes along with a wire shaped like a quarter circle, and tells you that the bead will reach the wire's end in* $T \approx 1.85\sqrt{(h/g)}$. *Proud to have found a wire shape that will make the bead reach the wire's end faster than either of the previous two shapes, he boldly posits that this wire shape must be the fastest possible way for the bead to get from* $(0, h)$ *to* $(h, 0)$*!*



Calculus students: Can you verify Galileo's claim with a calculation? In general, the time it takes for a bead to slide along a wire is given by the integral

$$T = \int_0^h \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{2g(h-y)}}\,dx, \tag{5}$$

Consider using a trigonometric substitution, if you're familiar with that technique. Note that

$$\frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}} \approx 1.854. \tag{6}$$

Unfortunately for Galileo, he was wrong! There is a special shape, discovered by Swiss mathematician brothers Johann and Jacob Bernoulli (1655-1705), called a *brachistochrone* (pronounced bruh-**kis**-tuh-krohn, from the Greek "brakhistos" meaning *shortest* and "kronos" meaning *time*), which will make the bead reach the end of the wire in the *shortest time* possible.

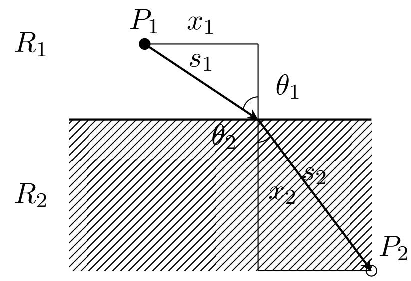
3 The Brachistochrone

Johann Bernoulli found a very clever derivation for the shape in 1697: clever, because it used physical reasoning to solve the mathematical puzzle. Pierre de Fermat (a French mathematician and judge living from 1607 to 1665) had formalized in 1662 the principle that light travels along the path of least time. Because light moves at different speeds through different materials, this tells us that light will bend when it moves from one material to another at an angle—a phenomenon referred to as *refraction.*

Refraction is described by an equation called *Snell's law*, which gives the solution to the problem of finding the path of least time for a special situation:

Snell's Law. Say there are two regions in space, a region R_1 where the speed of travel is v_1 and a region R_2 where the speed of travel is v_2, as illustrated below. The path of least time from a point P_1 in region R_1 to a point P_2 in region R_2 will satisfy the following relation:

$$\frac{x_1/s_1}{x_2/s_2} = \frac{v_1}{v_2} \tag{7}$$



If you're familiar with trigonometry, you know we have another way of saying this:

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}.$$

3.1. *Calculus students: See if you can derive Snell's law from the principle of least time using differentiation! Note that the total time will be*

$$T = \frac{s_1}{v_1} + \frac{s_2}{v_2}, \tag{8}$$

where s_1 and s_2 are the distances traveled in regions R_1 and R_2, respectively.

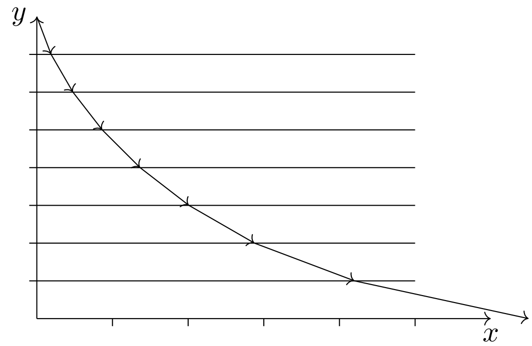
As an illustration of Snell's law, think of how you (or a dog fetching a stick) might go from some point on the shore to some point in the water. Chances are that you can run faster than you can swim, so you'll want to run a bit along the shore before you cross into the water, knowing you can cover the distance along the shoreline faster on foot. This is the principle of least time in action! Note that nothing about Snell's law is specific to light—that was just the context in which it was derived, and the context Bernoulli knew it from. In general, it's a principle for minimizing travel time. Light just happens to have no choice but to minimize time!

Equation 2 gives us a formula for the speed, v, of the ball as a function of height, y. Snell's law applies for two speed regions split by a clean boundary, but our function $v(y)$ is continuous, so how can we apply it? As in the case of finding the parabolic path of the ball, we can consider splitting y into small intervals, that, as we take more and more, will approximate the continuous function $v(y)$. At each one of these splits, the ball will be "refracted"—deflected—giving us a curve.

What we can note from Snell's law is that, across each boundary, this ratio of $\sin(\theta)/v$ is constant. Thus, we seek a curve where this ratio is constant at every point along it. Using the form of velocity given by equation (2), this means we want a curve where

$$\frac{x/s}{\sqrt{2g(h-y)}} = \text{constant} \tag{9}$$

at every point along it. What does such a curve look like? Here's an approximation:



This process of taking the limit of finer and finer intervals is at the heart of calculus, and it shows up in almost all problems of dynamics. There's a reason Newton discovered calculus [3] while working on the motion of the planets—he needed it to solve the problems of mechanics!
To learn more about Fermat's principle, Snell's law, and the brachistochrone, check out these videos:

- Snell's law: `https://www.youtube.com/watch?v=skvnj67YGmw`

- Brachistochrone derivation: `https://www.youtube.com/watch?v=Cld0p3a43fU`

- Brachistrochrone experiment: `https://www.youtube.com/watch?v=skvnj67YGmw`

[3] Alongside and independently from Leibniz

4 Additional Problems and Resources!

4.1. *Artie the ant, feeling rather queasy, stands near the rim of a turntable that is rotating at a constant rate c. He wants to walk directly to the center of the turntable at a constant rate a, where he suspects he might feel better. What is the trajectory of Artie as he makes his journey, as observed by an observer standing on the ground?*

Artie feels a lot less queasy standing near the center than he did on the rim. But, all of the sudden, some new excitement begins, as a pinball drops from the sky onto the turntable right next to where he's standing, with some initial velocity. What might the trajectory of the pinball look like for Artie in his rotating reference frame? What will this trajectory look like for the observer on the ground in their inertial reference frame?

4.2. *We're all familiar with the fact that circular wheels rolling along flat ground produce a smooth ride. However, is it possible for a square wheel to produce a smooth ride?*

Watch this video if you're interested in learning more about this problem: `https://https://www.youtube.com/watch?v=xGxSTzaID3k`*, or use this QR code:*

4.3. *A bicycle can be modeled as two point particles, one at the front wheel axle and one at the rear wheel axle. The two particles are constrained to be a fixed distance apart. However, there is an additional constraint at play, and it has to do with the fact that the front wheel can turn while the rear wheel cannot. Can you find a way to express this constraint mathematically? What are the degrees of freedom of the system (Hint: think about how many "inputs" you put into the bike when you ride it!).*

If the rear wheel has a trajectory that is a circle of radius a, and the distance between the wheels is b, what is the trajectory of the front wheel?

Can you show that the difference between the area swept out by the front wheel and the area swept out by the rear wheel will always be the same, so long as the the two tracks don't cross?

Watch this video if you're interested in learning more about this problem: `https://www.youtube.com/watch?v=l7bYY2U5ld8`*, or use this QR code:*