Linear Algebra 101

Prepared by Mark on April 20, 2023

Part 1: Notation and Terminology

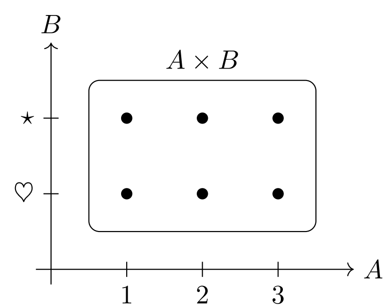
Definition 1:

- $\mathbb{R}$ is the set of all real numbers.
- $\mathbb{R}^+$ is the set of positive real numbers. Zero is not positive.
- $\mathbb{R}^+_0$ is the set of positive real numbers and zero.

Mathematicians are often inconsistent with their notation. Depending on the author, their mood, and the phase of the moon, $\mathbb{R}^+$ may or may not include zero. We will use the definitions above.

Definition 2:
Consider two sets A and B. The set $A \times B$ consists of all tuples (a, b) where $a \in A$ and $b \in B$.
For example, $\{1, 2, 3\} \times \{\heartsuit, \star\} = \{(1, \heartsuit), (1, \star), (2, \heartsuit), (2, \star), (3, \heartsuit), (3, \star)\}$
This is called the *cartesian product.*

You can think of this as placing the two sets "perpendicular" to one another:



Problem 1:
Let $A = \{0, 1\} \times \{0, 1\}$
Let $B = \{a, b\}$
What is $A \times B$?

Problem 2:
What is $\mathbb{R} \times \mathbb{R}$?
Hint: Use the "perpendicular" analogy

Definition 3:
$\mathbb{R}^n$ is the set of n-tuples of real numbers.
In English, this means that an element of $\mathbb{R}^n$ is a list of n real numbers:

Elements of $\mathbb{R}^2$ look like (a, b), where $a, b \in \mathbb{R}$.
Elements of $\mathbb{R}^5$ look like $(a_1, a_2, a_3, a_4, a_5)$, where $a_n \in \mathbb{R}$.
$\mathbb{R}^1$ and $\mathbb{R}$ are identical.

Note: $\mathbb{R}^2$ is pronounced "arrgh-two."

Intuitively, $\mathbb{R}^2$ forms a two-dimensional plane, and $\mathbb{R}^3$ forms a three-dimensional space.
$\mathbb{R}^n$ is hard to visualize when $n \geq 4$, but you are welcome to try.

Problem 3:
Convince yourself that $\mathbb{R} \times \mathbb{R}$ is $\mathbb{R}^2$.
What is $\mathbb{R}^2 \times \mathbb{R}$?

Part 2: Vectors

Definition 4:
Elements of $\mathbb{R}^n$ are often called *vectors*.
As you may already know, we have a few operations on vectors:

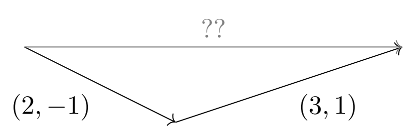
- Vector addition: $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$
- Scalar multiplication: $x \times [a_1, a_2] = [xa_1, xa_2]$.

The above examples are for $\mathbb{R}^2$, and each vector thus has two components.
These operations are similar for all other n.

Problem 4:
Compute the following or explain why you can't:

- $[1, 2, 3] - [1, 3, 4]$ Subtraction works just like addition.
- $4 \times [5, 2, 4]$
- $a + b$, where $a \in \mathbb{R}^5$ and $b \in \mathbb{R}^7$

Problem 5:
Consider $(2, -1)$ and $(3, 1)$ in $\mathbb{R}^2$.
Can you develop geometric intuition for their sum and difference?



Definition 5: Euclidean Norm
A *norm* on $\mathbb{R}^n$ is a map from $\mathbb{R}^n$ to $\mathbb{R}_0^+$
Usually, one thinks of a norm as a way of mesuring "length" in a vector space.
The norm of a vector v is written $||v||$.

We usually use the *Euclidean norm* when we work in $\mathbb{R}^n$.
If $v \in \mathbb{R}^n$, the Euclidean norm is defined as follows:
If $v = [v_1, v_2, ..., v_n]$,

$$||v|| = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$$

This is simply an application of the Pythagorean theorem.

Problem 6:
Compute the euclidean norm of

- $[2, 3]$
- $[-2, 1, -4, 2]$

Problem 7:
Show that $a \cdot a$ is $||a||^2$.

Part 3: Dot Products

Definition 6:
We can also define the *dot product* of two vectors.[1]
The dot product maps two elements of $\mathbb{R}^n$ to one element of $\mathbb{R}$:

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + ... + a_n b_n$$

Problem 8:
Compute $[2, 3, 4, 1] \cdot [2, 4, 10, 12]$

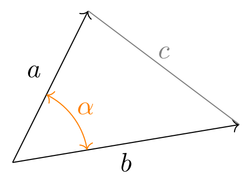
Problem 9:
Show that the dot product is

- Commutative
- Distributive $a \cdot (b + c) = a \cdot b + a \cdot c$
- Homogenous: $x(a \cdot b) = xa \cdot b = a \cdot xb$
  $x \in \mathbb{R}$, and a, b are vectors.
- Positive definite: $a \cdot a \geq 0$, with equality iff $a = 0$
  $a \in \mathbb{R}^n$, and 0 is the zero vector.

[1] **Bonus content. Feel free to skip.**
Formally, we would say that the dot product is a map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$. Why is this reasonable?

It's also worth noting that a function f from X to Y can be defined as a subset of $X \times Y$, where for all $x \in X$ there exists a unique $y \in Y$ so that $(x, y) \in f$. Try to make sense of this definition.

Problem 10:
Say you have two vectors, a and b. Show that $a \cdot b = ||a||\ ||b|| \cos(\alpha)$, where α is the angle between a and b.
Hint: What is c in terms of a and b?
Hint: The law of cosines is $a^2 + b^2 - 2ab\cos(\alpha) = c^2$
Hint: The length of a is $||a||$



Problem 11:
If a and b are perpendicular, what must $a \cdot b$ be? Is the converse true?

Part 4: Matrices

Definition 7:
A *matrix* is a two-dimensional array of numbers:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

The above matrix has two rows and three columns. It is thus a 2×3 matrix.

The order "first rows, then columns" is usually consistent in linear algebra.
If you look closely, you may also find it in the next definition.

Definition 8:
We can define the product of a matrix A and a vector v:

$$Av = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1a + 2b + 3c \\ 4a + 5b + 6c \end{bmatrix}$$

Note that each element of the resulting 2×1 matrix is the dot product of a row of A with v:

$$Av = \begin{bmatrix} \text{—}r_1\text{—} \\ \text{—}r_2\text{—} \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix} = \begin{bmatrix} r_1 \cdot v \\ r_2 \cdot v \end{bmatrix}$$

Naturally, a vector can only be multiplied by a matrix if the number of rows in the vector equals the number of columns in the matrix.

Problem 12:
Compute the following:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Problem 13:
Say you multiply a size-m vector v by an $m \times n$ matrix A.
What is the size of your result Av?

Definition 9:
We can also multiply a matrix by a matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 10 & 20 \\ 100 & 200 \end{bmatrix} = \begin{bmatrix} 210 & 420 \\ 430 & 860 \end{bmatrix}$$

Note each element of the resulting matrix is dot product of a row of A and a column of B:

$$AB = \begin{bmatrix} —r_1— \\ —r_2— \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} r_1 \cdot v_1 & r_1 \cdot v_2 \\ r_2 \cdot v_1 & r_2 \cdot v_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 10 & 20 \\ 100 & 200 \end{bmatrix} = \begin{bmatrix} 210 & 420 \\ 430 & 860 \end{bmatrix}$$

Problem 14:
Compute the following matrix product.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$$

Problem 15:
Compute the following matrix product or explain why you can't.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix}$$

Problem 16:
If A is an $m \times n$ matrix and B is a $p \times q$ matrix, when does the product AB exist?

Problem 17:
Is matrix multiplication commutative?
Does $AB = BA$ for all A, B?
You only need one counterexample to show this is false.

Definition 10:
Say we have a matrix A. The matrix A^T, pronounced "A-transpose", is created by turning rows of A into columns, and columns into rows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Problem 18:
Compute the following:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \qquad \begin{bmatrix} 1 \\ 3 \\ 3 \\ 7 \end{bmatrix}^T \qquad \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix}^T$$

The “transpose” operator is often used to write column vectors in a compact way.
Vertical arrays don’t look good in horizontal text.

Problem 19:
Consider the vectors $a = [1, 4, 3]^T$ and $b = [9, 1, 4]^T$

- Compute the dot product $a \cdot b$.
- Can you redefine the dot product using matrix multiplication?

As you may have noticed, a vector is a special case of a matrix.

Problem 20:
A *column vector* is an $m \times 1$ matrix.
A *row vector* is a $1 \times m$ matrix.
We usually use column vectors. Why?
Hint: How does vector-matrix multiplication work?

Part 5: Bonus

Problem 21:
Show that the euclidean norm satisfies the triangle inequalty:

$$||x + y|| \leq ||x|| + ||y||$$

Problem 22:
Show that the eucidean norm satisfies the reverse triangle inequality:

$$||x - y|| \geq |\ ||x|| - ||y||\ |$$

Problem 23:
Prove the Cauchy-Schwartz inequality:

$$||x \cdot y|| \leq ||x||\ ||y||$$