Learning to Count

Prepared by Mark on July 14, 2022

Part 1: Getting started

An ordered arrangement of objects is called a *permutation*. An unordered selection* of objects is called a *combination*. All the following problems involve permutations.

Problem 1:
How many different ways are there to rearrange the letters ABCDE?

Problem 2:
How many different ways are there to arrange the letters ABCDEFG...XYZ?
The answer is a *very* big number. You should not fully resolve your answer.
Hint: When you see a problem that's as big as this one, it's often wise to try and understand a simpler case first. Look at Problem 1 again, and try to create a general strategy.

*A "combination lock" cares about the order of its digits, so its name is inaccurate. Such an object is actually a *permutation* lock!

Definition 1:
The *factorial* of a positive integer x is $x \times (x-1) \times ... \times 1$. We denote this $x!$.
For example, $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$.

Problem 3:
Compute $\frac{10!}{8!}$

Problem 4:
Convince yourself that $(n+1)! = n! \times (n+1)$, and use this fact to show that $0! = 1$.

Problem 5:
How many ways are there to choose three student council officers from a class of 20 students?
How many ways are there to choose a president, a vice-president, and a treasurer from the same class?
Hint: You answers should be different. In which case does order matter?

Problem 6:
Say you have 4 red balls and 3 green balls. How many different ways can you arrange them on the table in font of you?

Problem 7:
How many *unique* anagrams can we create from the word CRESCENDO?

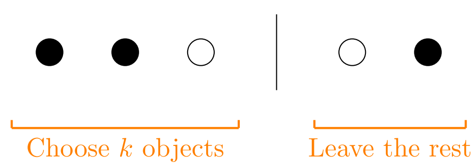
Problem 8:
Given the letters ABCDE, how many different three-letter words can we make without repeating letters?

Part 2: Permutations

It would be convenient to have a general tool for counting permutations. Let us try to create one. (Remember, permutations are *ordered* arrangements of objects.)
First, let's create a function ${}_nP_k$, which tells us how many k-object permutations we can choose from a group of n objects.

Problem 9:
What is ${}_5P_3$?
Hint: See Problem 8

"Choosing k items from n" is a lot like splitting our n objects into two groups: those we choose, and those we don't.



If we rearrange these, we get different permutations. How can we count them?

Problem 10:
Using the above diagram, create a formula for ${}_nP_k$.
Hint: We're counting *permutations*, so the order of items in the first group matters.

Part 3: Combinations

Now, let's count *combinations*.
Here, we care about *which* items we choose, but not *how* we choose them. We'll make a function ${}_nC_k$, or *"n choose k"*. This will tell us how many different ways we can choose k items from a set of n.

Problem 11:
Find an expression for ${}_nC_k$ by modifying your definition of ${}_nP_k$.

Usually, ${}_nC_k$ is written as $\binom{n}{k}$. This is also called the *binomial coefficient*.

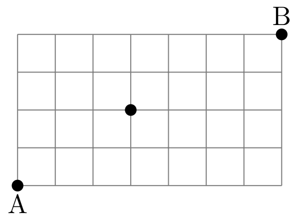
Part 4: Applications

Problem 12:
Use the meaning of ${}_nC_k$ to explain why ${}_nC_k = {}_nC_{n-k}$

Problem 13:
Use the formula of ${}_nC_k$ to explain why ${}_nC_k = {}_nC_{n-k}$

Problem 14:
How many ways can a class of 27 people be seated in 30 seats?

Problem 15:
The following is the map of a city. Each line is a one-way road, you can only drive up or right.
How many different paths can you take from A to B?
How many of them go through the center point?



Problem 16:
How many ways can you put 19 identical balls into 6 bins, so that no bin is empty?

Problem 17:
Given an exam with 4 problems, how many ways are there to assign positive point values to each problem so that the exam contains a total of 100 points?

Problem 18:
How many ways can we split the number 2016 into a sum of positive integers?

Problem 19:
A staircase must be built up a wall. It will start 4.5 meters away from the wall, which is 1.5 meters tall. The height of each step is exactly 30 centimeters. The width of each step must be an integer multiple of 50 centimeters. In how many ways can the staircase be constructed?

Part 5: Bonus problems

Problem 20:
A toy consists of a ring with 3 red beads and 7 blue beads on it. If two configurations of beads differ only by rotations and reflections, they are considered the same toy. How many different toys are there?

Problem 21:
At a math circle meeting, 10 students are given 10 problems. Any two students solved a different number of problems, and every problem is solved by the same number of students. Yan solved problems 1 through 5, but did not solve problems 6 through 9. Did he solve problem 10?

Problem 22:
A stressed-out student consumes at least one espresso every day of a particular year, drinking 500 overall. Prove that on some consecutive sequence of whole days the student drinks exactly 100 espressos. *Warning:* This problem is significantly harder than anything else in the handout.